Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Mathematics and Mechanics 71 (2007) 643–654
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Diffraction of elastic waves by an inclined crack in a layer☆

Ye.V. Glushkov, N.V. Glushkova, M.V. Golub

Krasnodar, Russia

Received 11 May 2006

Abstract

For the problem of the diffraction of normal modes by an inclined crack in an elastic layer, an integral equation with explicit representation of the Fourier symbol kernel is derived in the form of the product of matrices. The algorithm for calculating the wave fields, based on the analytical representations obtained, enables a rapid parametric analysis to be carried out of the influence of the size and orientation of the crack on the transmission of travelling waves. The influence of the inclination of the crack on the effects of resonance trapping and localization of wave energy, which were established previously for the case of a horizontal crack, is analysed.
© 2007 Elsevier Ltd. All rights reserved.

When solving problems of the diffraction of elastic waves by internal inhomogeneities, it is natural to take an integral approach, where the wave field is sought in the form of integral representations in terms of basic solutions (Green's matrix) and unknown functions at the boundary of the region examined (the method of boundary integral equations), or in the form of the superposition of solutions for the system of sources surrounding it (the method of fundamental solutions). Such representations identically satisfy the initial equations within the region, and thus approximation and discretization are carried out only at its boundary. Here, there is a considerable reduction in the number of unknowns (the dimension of the algebraic system) and consequently in the level of computing costs.

The methods of the integral approach occupy an intermediate position between direct numerical and asymptotic ray methods, combining their advantages. On the one hand, like the finite element method or the finite difference method, they give a quantitative solution at the points required, but with lower numerical costs; on the other hand, the asymptotic forms obtained from the integral representations (the contribution of poles, stationary points, etc.) give the same physically clear wave representations as ray methods, making it possible to analyse the wave structure of the solution, i.e. the distribution of the energy of oscillations between waves of different types.

To reduce the problems of the diffraction of elastic waves by a crack to integral equations, different approaches are used (see, for example, the review by Boström[1]). In the case of cracks in a boundless homogeneous space, equations of the Wiener–Hopf type, with a difference hypersingular kernel, arise, specified in the region occupied by the crack. For semi-bounded media (a half-space or layer), this type of equation is observed only for cracks parallel to the rectilinear boundaries of the medium (horizontal cracks), but for inclined cracks the classical approach, based on the use of a matrix of basic solutions for an elastic space, leads to integral equations specified on the entire surface of the waveguide and on the crack surfaces. Obtaining an integral equation specified only on the sides of the crack surfaces is made possible

☆ *Prikl. Mat. Mekh.* Vol. 71, No. 4, pp. 702–715, 2007.
E-mail address: evg@math.kubsu.ru (Ye.V. Glushkov).

0021-8928/$ – see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2007.09.011

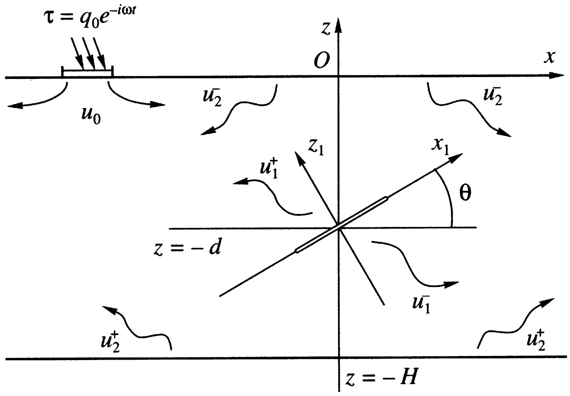

Fig. 1.

in this case by using Betti's reciprocity theorem (see, for example, Refs. 2,3). Thus, the problem of discretization on boundaries departing to infinity is removed.

In the present paper, a different technique for deriving the integral equation in the case of an inclined crack is given. Unlike the well-known approach,[2] it is based on the use of an integral representation of the wave field for a jump in displacements on an arbitrarily oriented internal area, which is constructed in the form of a combination of waves radiated by the crack and reflected from the boundaries of the medium, uniformly expressed by the Fourier symbols of Green's matrices of the half-space and layer. This representation identically satisfies the conditions on the outer boundaries of the layer, having a simpler and physically clear form.

Numerical algorithms based on the analytical representations obtained make it possible to carry out a rapid parametric analysis of the characteristics of the wave fields for different sizes and positions of the crack, including an analysis of the features of the transmission and reflection of Rayleigh–Lamb travelling waves. In this sense, the present paper is a natural continuation of investigations for the case of a horizontal crack;[4–6] a characteristic feature here is the presence of resonance frequencies at which trapping of wave energy occurs and there is sharp blocking of the waveguide, accompanied by a resonance increase in the stress intensity factors on the crack edges.

In an analysis of the influence of the angle of inclination of the crack on the resonance properties, the interesting effect of the waking of 'dormant' resonance poles is found. The presence of some of them does not show up in the case of one of the fundamental modes incident on a horizontal crack, but comes through with a small angle of inclination (see Section 5). Numerical results illustrating the effect of the energy transparency of a vertical crack when a higher normal mode appears are also given.

1. Mathematical model

In a plane formulation we consider the steady harmonic oscillations $\mathbf{u}(\exp)(-i\omega t)$, $\mathbf{u}=\{u_x, u_z\}$, of a free elastic waveguide of thickness H, occupying, in a Cartesian system of coordinates $\mathbf{x}=\{x, z\}$, a strip region $|x|<\infty$, $-H\le z\le 0$. (Here and below, the components of the corresponding column vector are given in curly brackets.) A strip crack (an infinitely thin rectilinear cut of width $2a$) makes an angle θ with the horizontal axis Ox; the surfaces crack do not come into contact with each other. The boundaries of the layer and the surfaces crack are assumed to be stress-free, with the exception, perhaps, of a local region of application of a specified load $\mathbf{q}_0 \exp(-i\omega t)$. As the source, it is also possible to consider the specified initial field $\mathbf{u}_0$, for example, for waves incident on the crack from infinity. In this case there is no surface load ($\mathbf{q}_0\equiv 0$).

It is assumed that, on the crack, the displacement field undergoes a discontinuity with an unknown jump $\mathbf{v}$:

$$\mathbf{v}(x_1) = C(\mathbf{u}|_{z_1=0^-} - \mathbf{u}|_{z_1=0^+}),\quad |x_1|\le a \tag{1.1}$$

Relation (1.1) is written not in global coordinates $\mathbf{x}$ but in crack-associated local coordinates $\mathbf{x}_1=\{x_1, z_1\}$ (Fig. 1). In these coordinates the crack occupies the segment $|x_1|\le a$, $z_1=0$.

The coordinates of a point in a global system ($\mathbf{x}$) and in a local system ($\mathbf{x}_1$) are connected by the relations

$$\begin{aligned}\mathbf{x}_1 &= C(\mathbf{x}-\mathbf{x}_c)\\ \mathbf{x} &= \mathbf{x}_c + C_1\mathbf{x}_1\end{aligned}, \quad C = \begin{Vmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{Vmatrix}, \quad C_1 = C^{-1} = C(-\theta) \tag{1.2}$$

in which $\mathbf{x}_c = \{0, -d\}$ are the global coordinates of the centre of the crack, d is the depth of the crack and $C(\theta)$ is the matrix of rotation by the angle θ. The matrix C in relation (1.1) converts the vector $\mathbf{u}$, specified in a global system, into a local system of coordinates.

By virtue of the linearity of the relations used, the harmonic factor $\exp(-i\omega t)$ will be omitted below, and, with regard to the complex amplitude of displacements $\mathbf{u}(\mathbf{x})$, the following boundary-value problem is formulated

$$(\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u} + \rho\omega^2\mathbf{u} = 0 \tag{1.3}$$

$$\boldsymbol{\tau}|_{z=0} = \begin{cases}\mathbf{q}_0(x), & |x-x_0| \le b\\ 0, & |x-x_0| > b\end{cases}, \quad \boldsymbol{\tau}|_{z=-H} = 0 \tag{1.4}$$

$$\boldsymbol{\tau}_1|_{z_1=0} = 0, \quad |x_1| \le a \tag{1.5}$$

where λ and μ are the Lamé constants, ρ is the density of the elastic medium, $\boldsymbol{\tau} = \{\tau_{xz}, \sigma_z\}$ is the stress vector on the horizontal surface and $\boldsymbol{\tau}_1$ is the stress vector on the crack surfaces in local coordinates. In the general case the stresses $\boldsymbol{\tau}_n$ on an area with unit normal $\mathbf{n}$ are related to the displacement field by the stress operator T_n:

$$\boldsymbol{\tau}_n = T_n\mathbf{u} \equiv \lambda\mathbf{n}\operatorname{div}\mathbf{u} + 2\mu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} + \mu(\mathbf{n}\times\operatorname{rot}\mathbf{u})$$

For horizontal surfaces with a normal $\mathbf{n} = \{0, 1\}$, the stress operator notation T_z is used below, similar to the operator in a local system – T_{z_1}. In this notation, the stress vectors under conditions (1.4) and (1.5) are connected with the displacements $\mathbf{u}$ by the following relations

$$\boldsymbol{\tau} = T_z\mathbf{u} \quad \boldsymbol{\tau}_1 = CT_\theta\mathbf{u}(\mathbf{x}) = T_{z_1}C\mathbf{u}(\mathbf{x}_1)$$

In addition to conditions (1.4) and (1.5), the radiation conditions at infinity ($x \to \pm\infty$), resulting from the limit absorption principle, are imposed on the solution $\mathbf{u}$. Within the framework of the technique used, they govern the direction in which real poles are bypassed by the integration loop Γ of the inverse Fourier transform F^{-1} and the choice of branches of the radicals $\sigma = \sqrt{\alpha^2 - \kappa_n^2}$ (see Ref. 7). In the special case where $H = \infty$, the layer degenerates into a half-plane; the last condition of (1.4) is replaced in this case by the radiation condition when $z \to -\infty$.

2. The structure of the wave field

The field $\mathbf{u}$ in a waveguide with a crack consists of the initial field $\mathbf{u}_0(\mathbf{x})$ excited in the waveguide without a crack by a specified load $\mathbf{q}_0$ or arriving from infinity, the crack-scattered field $\mathbf{u}_1(\mathbf{x}_1)$ and the field reflected from the boundaries of the layer $\mathbf{u}_2(\mathbf{x})$ (Fig. 1):

$$\mathbf{u} = \mathbf{u}_0 + C_1\mathbf{u}_1 + \mathbf{u}_2 \tag{2.1}$$

For convenience, the fields $\mathbf{u}_0$ and $\mathbf{u}_2$ are given in the global system $\mathbf{x}$, and the field $\mathbf{u}_1$ is given in the local system $\mathbf{x}_1$. Each of the vector functions $\mathbf{u}_n$ ($n = 0, 1, 2$) identically satisfies the initial Eq. (1.3) and the radiation conditions, while the boundary conditions split in the following way: the field $\mathbf{u}_0$ satisfies conditions (1.4), but continuously on the crack, and on the crack surfaces yields a non-zero stress field $\boldsymbol{\tau}_0 = CT_\theta\mathbf{u}_0|_{z_1=0}$, i.e. it does not satisfy conditions (1.1) and (1.5); the field $\mathbf{u}_1$ yields the required jump (1.1) but does not satisfy conditions (1.4) and (1.5) (the crack field in the entire space); the field $\mathbf{u}_2$ is introduced to correct of the conditions at the boundaries of the layer, i.e. it satisfies the following conditions stemming from conditions (1.4)

$$\boldsymbol{\tau}_2|_{z=0} = \mathbf{q}_2^+(x), \quad \boldsymbol{\tau}_2|_{z=-H} = \mathbf{q}_2^-(x),$$
$$\mathbf{q}_2^+ = -T_z C_1 \mathbf{u}_1|_{z=0}, \quad \mathbf{q}_2^- = -T_z C_1 \mathbf{u}_1|_{z=-H} \quad (2.2)$$

The stresses $\mathbf{q}_2^{\pm}$ are opposite in sign to the stresses on the boundaries of the layer induced by the field $\mathbf{u}_1$.

With this splitting, it only remains to satisfy the condition on the crack surfaces (1.5), which, taking into account the transformations carried out, we will write in the form

$$(T_{z_1}\mathbf{u}_1 + CT_\theta \mathbf{u}_2)|_{z_1=0} = -\boldsymbol{\tau}_0|_{z_1=0}, \quad |x_1| \le a \quad (2.3)$$

This condition is used to derive the matrix integral equation for the unknown jump $\mathbf{v}$ to which the solution of the initial boundary problem (1.3)–(1.5) reduces. Integral representations of the fields $\mathbf{u}_1$ and $\mathbf{u}_2$ in terms of $\mathbf{v}$ are constructed in advance.

3. Integral representations

The derivation of the required representations is based on well-known solutions for a half-plane in a layer in the form of the inverse Fourier transform

$$\mathbf{u}(\mathbf{x}) = \mathcal{F}^{-1}[\mathbf{U}] \equiv \frac{1}{2\pi}\int_\Gamma \mathbf{U}(\alpha, z)\exp(-i\alpha x)d\alpha \quad (3.1)$$

The Fourier symbol of the solution

$$\mathbf{U} = \mathcal{F}[\mathbf{u}] \equiv \int_{-\infty}^{\infty} \mathbf{u}(x, z)\exp(i\alpha x)dx$$

of these auxiliary problems is written in the form of the product of Fourier symbols of Green's matrix of the regions considered and the loads specified on their boundaries:

$$\mathbf{U}(\alpha, z) = K_\infty^{\pm}(\alpha, z)\mathbf{Q}(\alpha) \quad (3.2)$$

$$\mathbf{U}(\alpha, z) = K_H^-(\alpha, z)\mathbf{Q}^+(\alpha) + K_H^+(\alpha, z)\mathbf{Q}^-(\alpha) \quad (3.3)$$

where K_∞^+ and K_∞^- are the symbols of Green's matrices for the half-planes $z \ge 0$ and $z \le 0$ respectively, K_H^- is the symbol of Green's matrices for a layer of thickness H with non-zero stresses on only the upper face:

$$\boldsymbol{\tau}|_{z=0} = \mathbf{q}^+, \quad \boldsymbol{\tau}|_{z=-H} = 0$$

and K_H^+ is the symbol of Green's matrices for a layer of thickness H with non-zero stresses on only the lower face:

$$\boldsymbol{\tau}|_{z=0} = 0, \quad \boldsymbol{\tau}|_{z=-H} = \mathbf{q}^-, \quad \mathbf{Q}^{\pm} = \mathcal{F}[\mathbf{q}^{\pm}]$$

The technique for deriving Green's matrices is given, for example, in Ref. 7, and the explicit form of their elements in the notation used below is given in Ref. 5 (see also Glushkova, NV, Definition of and allowance for singular components in problems of elasticity theory. Doctorate Dissertation, 1.2.04, Rostov state University, 2000).

We will use representation (3.2) in the local system $\mathbf{x}_1$, taking as $\mathbf{q}$ the crack plane stresses $\mathbf{q}_1$ related to the field $\mathbf{u}_1$:

$$\mathbf{q}_1 = T_{z_1}\mathbf{u}_1|_{z_1=0}$$

In this case, relation (3.2) gives a representation for $\mathbf{U}_1$ when $z_1 \ge 0$ and $z_1 \le 0$, while for the symbol of the jump $\mathbf{V} = F_1[\mathbf{v}]$ from relations (1.1), (2.1) and (3.2) it follows ($\mathbf{Q}_1 = F_1[\mathbf{q}_1]$) that

$$\mathbf{V}(\beta) = [K_\infty^-(\beta, 0) - K_\infty^+(\beta, 0)]\mathbf{Q}_1(\beta) \quad (3.4)$$

or

$$\mathbf{Q}_1 = L_1\mathbf{V} \tag{3.5}$$

$$L_1(\beta) = [K_\infty^- - K_\infty^+]^{-1}\Big|_{z_1=0} = \frac{\Delta(\beta)}{\kappa_2^2}\begin{Vmatrix} 1/\sigma_2 & 0 \\ 0 & 1/\sigma_1 \end{Vmatrix} \tag{3.6}$$

$$\Delta(\beta) = 2\mu(\beta^2\sigma_1\sigma_2 - \gamma^4), \quad \gamma^2 = \beta^2 - \frac{1}{2}\kappa_2^2$$

$$\sigma_n = \sqrt{\beta^2 - \kappa_n^2}, \quad n = 1, 2, \quad \kappa_1^2 = \frac{\rho\omega^2}{\lambda + 2\mu}, \quad \kappa_2^2 = \frac{\rho\omega^2}{\mu}$$

where β or later β_1 is the parameter of the Fourier transform with respect to x_1. Note that, of the terms in relation (2.1), a contribution to $\mathbf{v}$ is made only by $\mathbf{u}_1$ (in relations (3.4) and (3.5) C_1 does not occur, since $C \cdot C_1 = I$ is an identity matrix). As a result, for the Fourier symbol of the field $\mathbf{u}_1$ in local coordinates we have

$$\mathbf{U}_1(\beta, z_1) = K_\infty^\pm(\beta, z_1)L_1(\beta)\mathbf{V}(\beta) \tag{3.7}$$

By virtue of conditions (2.2), to obtain the expression of $\mathbf{U}_2(\alpha, z)$ in terms of $\mathbf{V}$, it is now sufficient to replace $\mathbf{Q}^\pm$ in Eq. (3.3) by the values $\mathbf{Q}_2^\pm$ expressed in terms of $\mathbf{U}_1$ in the form of relation (3.7). However, $\mathbf{Q}_2^\pm$ must be specified in the form of the Fourier transform with respect to x in the global system $\mathbf{x}$, whereas $\mathbf{U}_1 = F_1[\mathbf{u}_1]$ is the transform with respect to x_1. The non-parallel nature of these axes is the main obstacle to deriving convenient representations. If the inverse transform F_1^{-1} is applied directly to Eq. (3.7), and then the rotation operator C_1, the stress operator T_2 and the operator of the Fourier transform with respect to x, then, instead of an explicit matrix relation connecting $\mathbf{Q}_2^\pm$ with $\mathbf{V}$, we obtain a cumbersome expression with double integration.

The required matrix relation can be obtained if the approach proposed for an inclined crack in a half-space[8] is used. The idea consists in rotating the system of coordinates not only in the initial space but also in the space of the parameters of the complete (for all spatial coordinates) Fourier transform $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2\}$, $(\alpha_1 \equiv \alpha)$. The components of the Fourier symbol, depending only on the length of the vector $\boldsymbol{\alpha}$, and the scalar product $(\boldsymbol{\alpha}, \mathbf{x}) = \alpha_1 x + \alpha_2 z$ are invariant under such rotation, which makes subsequent calculations very much easier, enabling, in particular, the residue integrals arising here to be taken.

To obtain the complete transform $\hat{\mathbf{U}}_1(\boldsymbol{\beta})$, $\boldsymbol{\beta} = \{\beta_1, \beta_2\}$ $(\beta_1 \equiv \beta)$, in the local system $\mathbf{x}_1$, we will apply to $\mathbf{U}_1(\beta_1, z_1)$ the transformation with respect to z_1: $\hat{\mathbf{U}}_1 = F_{z_1}[\mathbf{U}_1]$. The dependence on z_1 occurs in $K_\infty^\pm$ only via the exponent $e_n = \exp(-\sigma_n|z_1|)$, and therefore the integrals arising here are taken in explicit form:

$$\mathcal{F}_{z_1}[e_n] = \frac{2\sigma_n}{\delta_n(\boldsymbol{\beta})}, \quad \mathcal{F}_{z_1}[\operatorname{sign} z_1 e_n] = \frac{2i\beta_2}{\delta_n(\delta)}$$

$$\delta_n(\boldsymbol{\beta}) = \beta_2^2 + \sigma_n^2 = |\boldsymbol{\beta}|^2 - \kappa_n^2, \quad n = 1, 2$$

As a result

$$\hat{\mathbf{U}}_1(\boldsymbol{\beta}) = \hat{K}(\boldsymbol{\beta})\mathbf{V}(\beta_1), \quad \hat{K}(\boldsymbol{\beta}) = \frac{2i}{\kappa_2^2}\sum_{n=1}^{2}\frac{\hat{K}_n(\boldsymbol{\beta})}{\delta_n(\boldsymbol{\beta})}$$

$$\hat{K}_1 = \begin{Vmatrix} \beta_1^2\beta_2 & \beta_1\gamma^2 \\ \beta_1\sigma_1^2 & -\beta_2\gamma^2 \end{Vmatrix}, \quad \hat{K}_2 = \begin{Vmatrix} -\beta_2\gamma^2 & -\beta_1\sigma_2^2 \\ -\beta_1\gamma^2 & \beta_2 \end{Vmatrix} \tag{3.8}$$

The conversion of $\mathbf{u}_1$ into the global system $\mathbf{x}$ requires the replacement of relations (1.2) in the inverse transform

$$\mathbf{u}_1(\mathbf{x}_1) = \frac{1}{(2\pi)^2}\int_{\Gamma_1}\int_{\Gamma_2}\hat{\mathbf{U}}_1(\boldsymbol{\beta})\exp(-i(\boldsymbol{\beta},\mathbf{x}_1))d\beta_1 d\beta_2, \quad (\boldsymbol{\beta},\mathbf{x}_1) = \beta_1 x_1 + \beta_2 z_1 \tag{3.9}$$

If, at the same time, rotation in the plane of integration $\boldsymbol{\beta}$ is carried out using the same rotation matrix C: $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{\alpha}) = C\boldsymbol{\alpha}$, then these replacements do not alter the structure of the exponent in relation (3.9):

$$(\boldsymbol{\beta},\mathbf{x}_1) = (C\boldsymbol{\alpha}, C(\mathbf{x}-\mathbf{x}_c)) = (\boldsymbol{\alpha}, C^{-1}C(\mathbf{x}-\mathbf{x}_c)) = (\boldsymbol{\alpha},\mathbf{x}) - (\boldsymbol{\alpha},\mathbf{x}_c)$$

or the length of the vector $\boldsymbol{\beta}$: $|\boldsymbol{\beta}| = |C\boldsymbol{\alpha}| = \boldsymbol{\alpha}$. In particular, in the second equation of system (3.8), $\delta_n(\boldsymbol{\beta}) = \delta_n(\alpha) = |\alpha^2| - \kappa_n^2$, i.e. the polar factors $|\alpha| = \kappa_n$ of the integrand are also not changed.

Thus, for the stress vector $\mathbf{q}_2^+(x) = -T_z C_1 \mathbf{u}_1|_{z=0}$ we have

$$\begin{aligned}\mathbf{q}_2^+(x) &= -\frac{1}{(2\pi)^2}\int_{\Gamma_1}\int_{\Gamma_2} T_z(\boldsymbol{\alpha})C_1\hat{K}(\boldsymbol{\beta}(\boldsymbol{\alpha}))\mathbf{V}(\beta_1(\boldsymbol{\alpha}))\exp(-i\alpha_2\pi d)\exp(-i\alpha_1 x)d\alpha_1 d\alpha_2 \\ T_z(\boldsymbol{\alpha}) &= \mathcal{F}_{xz}[T_z] = -i\left\|\begin{matrix}\mu\alpha_2 & \mu\alpha_1 \\ \lambda\alpha_1 & (\lambda+2\mu)\alpha_2\end{matrix}\right\|\end{aligned} \tag{3.10}$$

where T_z is the Fourier symbol of the stress operator in the space of the double transform with respect to x and z. Exponential decay of the integrand in the lower half-space of the complex plane α_2 (through the exponent $\exp(-i\alpha_2 d)$, $d > 0$) enables as to apply Jordan's lemma for the integral with respect to α_2, closing the loop Γ_2 in the lower half-plane, and then to replace it with the contribution of residues at the poles falling within the closed loop

$$\alpha_2 = \pm i\sigma_n, \quad \sigma_n(\alpha_1) = \sqrt{\alpha_1^2 - \kappa_n^2}$$

The branches of these radicals (as above, for $\sigma_n(\beta_1)$) are fixed by the condition $\mathrm{Re}\sigma_n \geq 0$, $\mathrm{Im}\sigma_n \leq 0$ for real α_1 and κ_n. The limit absorption principle gives the poles a bypass by the loop Γ_2: $i\sigma_n$ above, and $i\sigma_n$ below (when they reach the real axis for pure imaginary values of σ_n). Therefore, the residues for $\mathbf{q}_2^+$ are taken at the poles $\alpha_2 = -i\sigma_n$. With similar considerations for $\mathbf{q}_2^-(x)$ with $z = -H$, the exponent $\exp(i\alpha_2(H - \mathrm{d}))$ arises, determining the closure of Γ_2 in the upper half-plane and the contribution of the residues at the poles $\alpha_2 = i\sigma_n$.

Obviously, the integrands subsequently obtained in the inverse Fourier integrals with respect to α_1 will also be the required explicit representations for $\mathbf{Q}_2^\pm(\alpha)$:

$$\begin{aligned}\mathbf{Q}_2^\pm(\alpha) &= -\sum_{n=1}^{2} T_n^\pm(\alpha)C_1 K_n^\pm(\alpha)\mathbf{V}_n^\pm(\alpha) \\ T_n^\pm &= T_z(\alpha)|_{\alpha_2 = \mp i\sigma_n}, \quad K_n^\pm = \frac{\hat{K}_n(\boldsymbol{\beta}(\boldsymbol{\alpha}))|_{\alpha_2=\mp i\sigma_n}}{2\sigma_n}\begin{cases}\exp(-\sigma_n d) & \text{for } K_n^+ \\ \exp(-\sigma_n(H-d)) & \text{for } K_n^-\end{cases} \\ \mathbf{V}_n^\pm &= \mathbf{V}(\beta_1(\boldsymbol{\alpha}))|_{\alpha_2=\mp i\sigma_n} = \int_{-a}^{a}\mathbf{v}(\xi_1)\exp((i\cos\theta\alpha_1 \pm \sin\theta\sigma_n)\xi_1)d\xi_1\end{aligned} \tag{3.11}$$

Thus, the Fourier symbol of the field reflected from the boundaries of the layer

$$\mathbf{U}_2(\alpha, z) = K_H^-(\alpha, z)\mathbf{Q}_2^+(\alpha) + K_H^+(\alpha, z)\mathbf{Q}_2^-(\alpha) \tag{3.12}$$

as with $\mathbf{Q}_1$ and $\mathbf{U}_1$ of the form of Eqs. (3.5) and (3.7), is also expressed in terms of $\mathbf{V}$, which enables us to derive an integral equation for $\mathbf{v}$.

4. Integral equation

We will substitute the representations (3.7), (3.11) and (3.12) obtained into the condition (2.3) in the form of inverse Fourier transforms. Taking into account that

$$T_z \mathbf{u}_1|_{z_1=0} = \mathbf{q}_1 = \mathcal{F}_1^{-1}[L_1 \mathbf{V}]$$

and, changing the order of integration, we arrive at a matrix integral equation that, in operator notation, has the form

$$\mathcal{L}\mathbf{v} \equiv \mathcal{L}_1\mathbf{v} + \mathcal{L}_2\mathbf{v} = \mathbf{f}_0, \quad |x_1| < a \tag{4.1}$$

where

$$\mathcal{L}_1\mathbf{v} = \int_{-a}^{a} l_1(x_1 - \xi_1)\mathbf{v}(\xi_1)d\xi_1, \quad l_1 = \mathcal{F}_1^{-1}[L_1] \ (L_1 \text{ see } (3.6))$$

$$\mathcal{L}_2\mathbf{v} = \int_{-a}^{a} l_2(x_1, \xi_1)\mathbf{v}(\xi_1)d\xi_1$$

$$l_2(x_1, \xi_1) = \frac{1}{2\pi}\int_{\Gamma} L_2(\alpha, x_1, \xi_1)\exp(-i\alpha(x_1 - \xi_1)\cos\theta)d\alpha$$

$$L_2 = CT_\theta\left(\alpha, \frac{\partial}{\partial z}\right)[K_H^-(\alpha, z)P^+(\alpha, \xi_1) + K_H^+(\alpha, z)P^-(\alpha, \xi_1)]_{z=-d+x_1\sin\theta} \tag{4.2}$$

$$P^\pm = -\sum_{n=1}^{2} T_n^\pm(\alpha)C_1K_n^\pm(\alpha)\exp(\pm\xi_1\sigma_n(\alpha)\sin\theta)$$

$$T_\theta\left(\alpha, \frac{\partial}{\partial z}\right) = \mathcal{F}[T_\theta], \quad \mathbf{f}_0(x_1) = -\boldsymbol{\tau}_0|_{z_1=0}$$

The matrix operator $T_\theta(\alpha, \partial/\partial z)$ is obtained from T_θ by replacing the operator of the derivative $\partial/\partial x$ by $-i\alpha$. The remaining operators $\partial/\partial z$ act on $K_H^\mp(\alpha, z)$; the elements of the matrices $\partial K_H^\mp/\partial z$ are written in explicit form.

The operator L_1 is a well-known hypersingular integral operator for the problem of diffraction by a crack in boundless space. The hypersingularity

$$l_1(x_1 - \xi_1) = O(|x_1 - \xi_1|^{-2}) \text{ при } \xi_1 \to x_1$$

is due to the exponential growth of the symbol $L_1(\alpha) = O(\alpha)$, $\alpha \to \infty$. By virtue of the diagonality of L_1, such a problem breaks down into two independent problems in the normal and shear components of the jump **v**.

The operator L_2 describes the influence of waves reflected from the boundaries of the waveguide. If the crack does not touch the surface, the integral over α in the representation of l_2 converges for all x_1, ξ_1 on account of exponentially decaying factors in the representation of L_2. In this case the kernel l_2 is smooth for both variables. In the case of contact, such decay does not occur when $x_1 = \xi_1 = a$ (or $-a$), which leads to a singularity of the elements of l_2 at this point. The singularity of l_2, however, is weaker than the hypersingularity of l_1. When L_2 is present, the matrix of the overall operator L becomes filled, and therefore the matrix integral Eq. (4.1) does not break down into two scalars.

The widely used method for the numerical solution of one-dimensional integral equations involves replacing the integral operator with a quadrature sum with subsequent determination of the values of the unknown at the approximation holds from the linear algebraic system, arising, for example, from collocation conditions. With strong singularity of the kernel, such an approach requires strict observation of a certain arrangement of approximation and collocation nodes that ensures regularization of the system and a stable and rapid convergence of the numerical solution.[9]

Another popular method, when solving problems of diffraction by a crack, for overcoming problems associated with kernel hypersingularity is based on the use of the property of orthogonality of certain polynomials with the weight generated by the singular part L_s of the operator $L = L_s + L_c$:

$$(\mathcal{L}_s\varphi_k, \psi_j)_{L_2} = d_j\delta_{jk}$$

where δ_{jk} is the Kronecker delta, and L_c is the smooth part of the operator L.[10–12] The expansion, by such a system, of the basis functions φ_k

$$\mathbf{v} \approx \mathbf{v}_N = \sum_{k=1}^{N} \mathbf{c}_k\varphi_k(x_1) \tag{4.3}$$

with subsequent determination of the coefficients $\mathbf{c}_k$ using Bubnov's scheme (the orthogonality of the discrepancy $L\mathbf{v}_N - \mathbf{f}_0$ to the system of projectors ψ_j) leads to a linear algebraic system of the second kind in $\mathbf{c} = \{\mathbf{c}_1, \ldots, \mathbf{c}_N\}$:

$$A\mathbf{c} \equiv (D + A_c)\mathbf{c} = \mathbf{f} \tag{4.4}$$

Hear

$$A = [a_{jk}]_{jk=1}^{N}, \quad D = \operatorname{diag}[d_j]_{j=1}^{N}, \quad A_c = [a_{c,jk}]_{jk=1}^{N}, \quad \mathbf{f} = \{\mathbf{f}_1, \ldots, \mathbf{f}_N\}$$

$a_{jk} = (L\varphi_j, \psi_k)_{L_2}$ and $a_{c,jk} = (L_c\varphi_j, \psi_k)_{L_2}$ are 2×2 partitioned matrices and $\mathbf{f}_j = (\mathbf{f}, \psi_j)_{L_2}$ are vectors of length 2. The elements d_j are written from the conditions of orthogonality in explicit form, whereas the definition of the elements of the matrix A_c does not require the evaluation of singular integrals.

However, for a regularization ensuring numerical stability of the solution of the algebraic system, the singular component does not have to be isolated in explicit form. It is sufficient, as the basis φ_k, to select a system of functions strictly allowing for the nature of the behaviour of the solution at the edges of the segment $-a \le x_1 \le a$. The well-known root feature of stresses in the vicinity of the crack edges for the required jump in displacements $\mathbf{v}$ corresponds to a root tendency to zero:

$$\mathbf{v}(x) \sim \mathbf{v}_0^{\pm}\sqrt{a \mp x_1}, \quad x_1 \to \pm a$$

Therefore, the natural basis here comprises Chebyshev polynomials of the second kind $U_k(x)$, orthogonal with weight $\sqrt{1-x^2}$:

$$\varphi_k(x_1) = U_k(x_1/a)\sqrt{1-(x_1/a)^2}$$

The form of the projectors ψ_j is not of such great importance; in particular, $\psi_j(x_1) = \varphi_j(x_1)$ were used. Note that the stress intensity factors (SIF)

$$\mathbf{k}^{\pm} : \boldsymbol{\tau}_1(x_1) \sim \mathbf{k}^{\pm}/\sqrt{a \mp x_1}, \quad x_1 \to \pm a$$

are uniquely expressed in terms of $v_0^{\pm}$.[2]

At first glance it appears that the determination of the matrix elements of the system a_{jk} requires numerical integration of triple integrals. However, in the integral representation of the kernels l_1 and l_2, the dependence on x_1 and ξ_1 appears only through the exponents (see relation (4.2)), and therefore the given integrals are expressed in terms of the Fourier transform of the basis functions $\Phi_k = F_1[\varphi_k]$ and $\Psi_j = F[\psi_j]$. For the selected polynomials they are written in the explicit form[13]

$$\Phi_k(\alpha) = \Psi_j(\alpha) = \pi a i^k (k+1) J_{k+1}(a\alpha)/\alpha$$

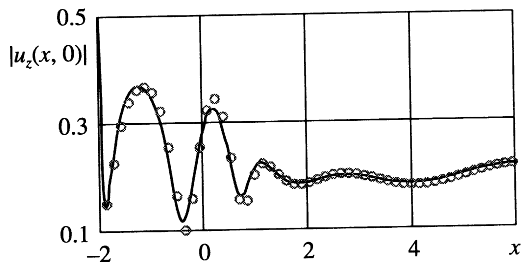

Fig. 2.

As a result, we arrive at representations in the form of single integrals with respect to α:

$$a_{jk} = a_{1,jk} + a_{2,jk}$$
$$a_{1,jk} = \frac{1}{2\pi}\int_\Gamma L_1(\alpha)\Phi_k(\alpha)\Psi_j(-\alpha)d\alpha, \quad a_{2,jk} = \frac{1}{2\pi}\int_\Gamma \tilde{L}_2(\alpha,\theta)d\alpha \tag{4.5}$$

$\tilde{L}_2(\alpha, \theta)$ has the form (4.2) in which, at the position of the exponents $\exp(\mp\xi_1\sigma_n \sin\theta)$ in the matrix $P^{\pm}$, the functions $\Phi_k(\alpha\cos\theta \mp i\sigma_n \sin\theta)$ occur, and, at the position of the exponents $\exp(\mp\sigma_n z)$, the functions $\Psi_j(-\alpha\cos\theta \mp i\sigma_n \sin\theta)\exp(\mp d\sigma_n)$.

The contour integrals with respect to α, similar to the integrals (4.5), traditionally arise when solving dynamic problems within the framework of the integral approach (see, for example, Refs 4–8), and therefore methods for their numerical integration are now well developed. The main problem here is the slow convergence of the integrals at infinity on account of the growth of $L_1(\alpha)$ when $\alpha \to \infty$ (and the non-exponential decay of L_2 when the crack reaches the surface). Acceleration of the convergence is achieved by isolating and integrating the slowly converging components in explicit form, which is equivalent to the above-mentioned isolation of the singular part L_s of the operator L.

Note that in Bubnov's scheme the presence of the functions Φ_k and Ψ_j ensures better convergence of the integrals than when calculating the values of the kernels l_1 and l_2 at fixed points, which is necessary in the collocation method. Therefore, in spite of the apparent simplicity, the use of the collocation method to solve Eq. (4.1) requires no less expenditure than does the implementation of Bubnov's scheme.

5. Numerical analysis

The reliability of the results obtained was monitored by a numerical check of the boundary conditions and energy balance, and also by comparison with the numerical results of other authors. For an inclined crack in a half-space, examples of comparison with the results of Ref. 2 are given in Ref. 8. For a crack in a layer, results[14] obtained by the strip-element method were used as the control. The small circles in Fig. 2 give the values of the amplitude of vertical oscillations of the surface $|u_z(x, 0)|$, excited by the load $\mathbf{q}_0(x) = \{0, \delta(x+2)\}$ (see Eq. (1.4)), in a layer with an inclined crack ($\theta = 5.7°$, $a = 1.005$, $d = 0.5$, $\omega = 1.57$), obtained by means of the integral representations derived in the present paper (by solving integral Eq. (4.1)); the continuous curve is the graph obtained by the finite element method.[14]

Here and below, the results are given in dimensionless form in units expressed in terms of quantities H, υ_s and ρ. Thereby, the dimensionless angular velocity $\omega = 2\pi f H/\upsilon_s$, where f, H and υ_s are the dimensional frequency, layer thickness and velocity of the S waves in the layer, and all the linear dimensions are related to H.

Let us consider how the inclination of the crack, θ, affects the behaviour of the resonance poles ω_n (the spectral points of the integral operator L) closest to the real axis. Numerically they are approximated by roots of the characteristic equation $\det A(\omega) = 0$, where A is the matrix of system (4.4). The values of ω_n given in the Table 1 for a mean crack of unit half-width ($d = 1/2$, $a = 1$) with different angles of inclination indicate that $\mathrm{Re}\,\omega_n$ depends weakly on θ, whereas all $|\mathrm{Im}\,\omega_n|$, apart from $n = 3$, increase as θ increases, i.e. as inclination of the crack increases they move further away from the real axis.

Earlier it was established that, in the case of a horizontal crack at frequencies $\omega \approx \mathrm{Re}\,\omega_n$, sharp trapping of the traveling wave energy is possible, leading to blocking the waveguide and an increases in the SIF.[4–6] However, for a

Table 1

θ	ω_1	ω_2	ω_3	ω_4	ω_5
0°	0.734–0.0003*i*	1.755–0.003*i*	2.097–0.104*i*	2.944–0.0007*i*	4.190–0.025*i*
4.5°	0.737–0.011*i*	1.760–0.014*i*	2.101–0.103*i*	2.954–0.005*i*	4.205–0.032*i*
9°	0.756–0.052*i*	1.774–0.050*i*	2.116–0.100*i*	2.983–0.019*i*	4.249–0.056*i*
15°		1.786–0.168*i*	2.152–0.095*i*	2.051–0.054*i*	4.336–0.138*i*

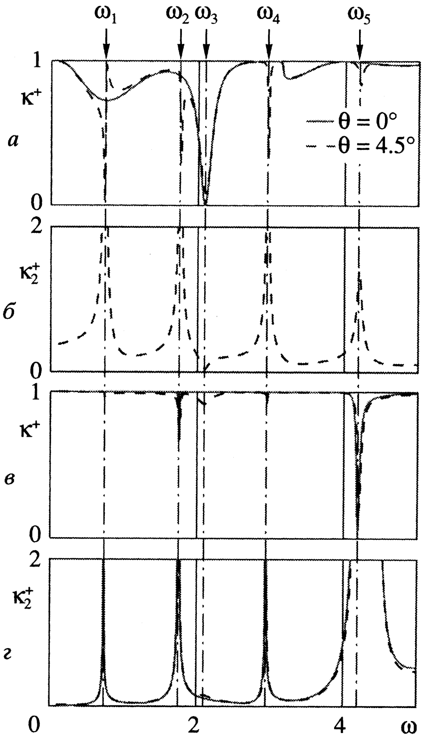

Fig. 3.

middle crack, the appearance of these effects depends on the type of travelling wave. Thus, when a zero antisymmetric mode a_0 is incident on the above-examined horizontal crack (the continuous curve in Fig. 3), blocking (i.e. a sharp reduction in the transmission coefficient κ^+) is observed only in the vicinity of the third pole ω_3 (see Fig. 3a). However, when a zero symmetric mode s_0 is incident on the crack, conversely, blocking occurs at all ω_n apart from ω_3 (Fig. 3c) (for ω_1 on the scale of the given figure it is not noticeable, since it occurs in a very narrow frequency range). Here and below, κ^+ is defined as the ratio of the energy of passing waves to the energy of the travelling wave. Thus, the set of poles can be divided into two groups: in the first group ω_3, and in the second group all the others. It is interesting that resonance growth of the SIF only occurs at the frequencies of the second group, for both a_0 and s_0 incident on the crack (see Figs. 3b and d respectively).

Of equal interest is the effect observed in the case of an inclined crack. At $\theta = 4.5°$ (the dashed curve in Fig. 3), all ‘dormant’ before this poles of the second group give resonance blocking the mode a_0 (Fig. 3a). At the same time, in this case also, ω_3 does not appear at all on the SIF graphs (Figs. 3b and d).

More information about the dependence of the transmission factor κ^+ and the stress intensity factor k_2^+ on the angle of inclination θ and the frequency ω is given by the surfaces $\kappa^+(\theta, \omega)$ and $k_2^+(\theta, \omega)$ constructed for the same travelling fundamental modes a_0 and s_0. In Fig. 4, these surfaces are shown by the level lines and by the grey colour. The dark zones in Figs. 4a and c show the waveguide blocking mode, and those in Figs. 4b and d show the growth in the SIF. The graphs in Fig. 3 show the cross-sections of the corresponding surfaces of Fig. 4 along the lines $\theta = 0°$ and $\theta = 4.5°$. The results given in Figs 4a and b indicate, in particular, that the resonance blocking on account of pole ω_3, as with the

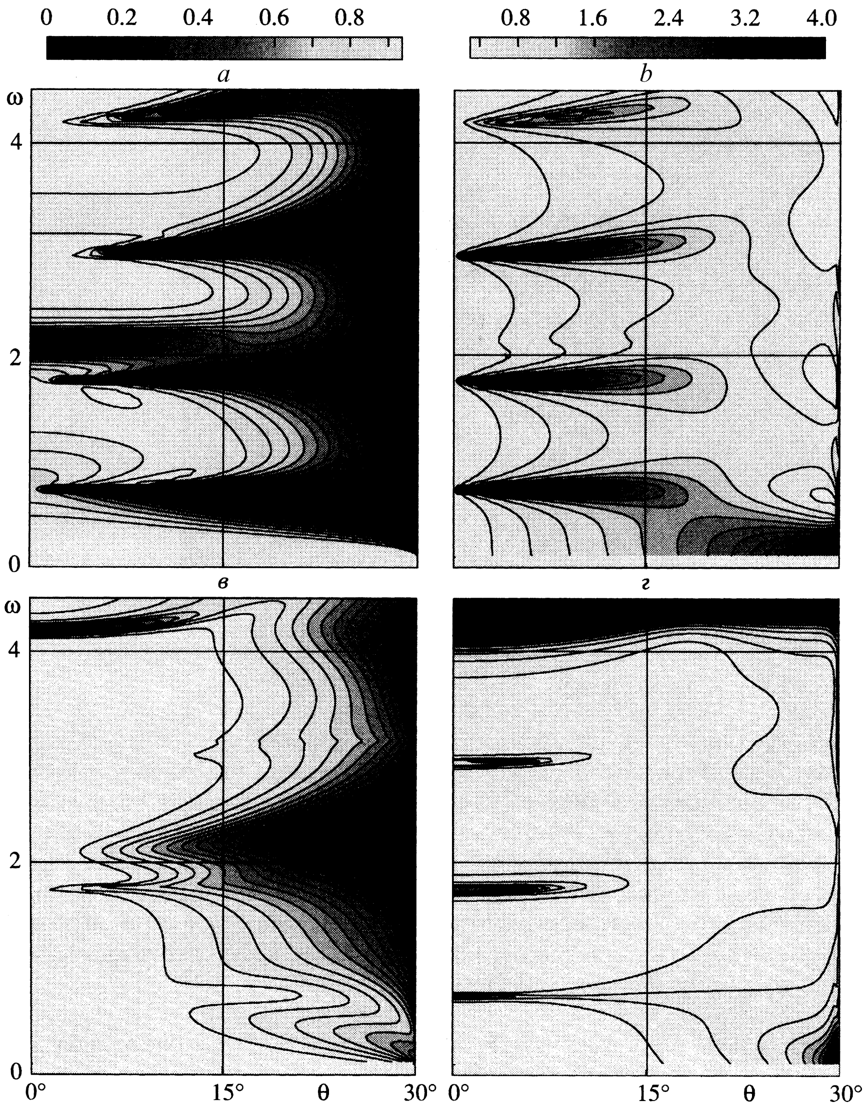

Fig. 4.

growth in the SIF on account of poles of the second set, is traced roughly up to $\theta = 15°$. At the same time, the blocking properties of the second set, which are not immediately evident, are retained up to $\theta = 30°$, when the cross-section of the waveguide is covered entirely by the inclined crack ($a \sin\theta = d$).

The continuous dark area on the right-hand side of Figs. 4a and c corresponds to the situation where the crack covers the cross-section of the waveguide almost completely. In some cases, however, the crack may remain transparent for the passage of the wave energy, even with partitioning of up to 95% of the layer thickness. By way of example, Fig. 5 gives the surface $\kappa^+(a, \omega)$ for a vertical crack ($\theta = 90°$, $d = 1/2$) in the case when mode a_0 is incident on it. In the range $0 < \omega < \pi$, where only two fundamental modes a_0 and s_0 can be excited in the layer, an increase both in the length of the crack and in the frequency ω leads to a monotonic reduction in κ^+. However, when the next highest mode appears with $\omega > \pi$, the crack practically ceases to block the energy flux, although, undoubtedly, its redistribution between diffraction-excited modes occurs. The effect of energy transparency appears in Fig. 5 in the form of a white horizontal band at the level $\omega \approx 3.2$. The dark spot at its origin ($a = 0.2$) is the result of resonance blocking at a comparatively small crack size.

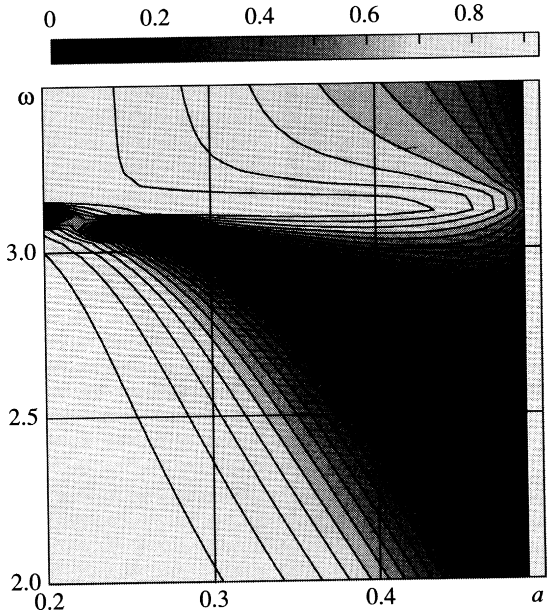

Fig. 5.

Acknowledgement

This research was supported by the Russian Foundation for Basic Research (04-01-00801, 06-01-96607) and the International Association for Cooperation and Collaboration with Scientists from the New Independent States of the Former Soviet Union (INTAS 05-1000008-7979).

References

1. Boström A. Review of hypersingular integral equation method for crack scattering and application to modeling of ultrasonic nondestructive evaluation. *Appl Mech Rev* 2003;**56**(4):383–405.
2. Van der Hijden JHMT, Neerhoff FL. Diffraction of elastic waves by a sub-surface crack (in-plane motion). *J Acoust Soc Am* 1984;**75**(6):1694–704.
3. Vatul'yan AO. Boundary integral equations of the first kind in dynamic problems of the anisotropic theory of elasticity. *Dokl Ross Akad Nauk* 1993;**333**(3):312–4.
4. Glushkov YeV, Glushkova NV, Shapar' YeM. Blocking of a Rayleigh wave by a near-surface crack. *Dokl Ross Akad Nauk* 2004;**398**(6):764–70.
5. Glushkov YeV, Glushkova NV, Golub MV. Blocking of travelling waves and localization of the energy of elastic vibrations in diffraction by a crack. *Akust Zhurnal* 2006;**52**(3):314–25.
6. Glushkov E, Glushkova N, Golub M, Boström A. Natural resonance frequencies, wave blocking, and energy localization in an elastic half-space and waveguide with a crack. *J Acoust Soc Am* 2006;**119**(6):3589–98.
7. Babeshko VA, Glushkov YeV, Zinchenko ZhF. *Dynamics of Inhomogeneous Linear Elastic Media*. Moscow: Nauka; 1989.
8. Glushkov E, Glushkova N, Ekhlakov A, Shapar E. An analytically based computer model for surface measurements in ultrasonic crack detection. *Wave Motion* 2006;**43**(6):458–73.
9. Lifanov IK. *The Method of Singular Integral Equations and Numerical Experiment (in Mathematical Physics, Aerodynamics, the Theory of Elasticity, and Wave Diffraction)*. Moscow: TOO "Yanus"; 1995.
10. Aleksandrov VM, Smetanin BI, Sobol' BV. *Thin Stress Concentrators in Elastic Bodies*. Moscow: Fizmatlit; 1993.
11. Popov GYa. *Concentration of Elastic Stresses near Punches, Cuts, Thin Inclusions, and Reinforcements*. Moscow: Nauka; 1980.
12. Krenk S, Schmidt H. Elastic wave scattering by a circular crack. *Phil Trans Roy Soc London* 1982;**A308**(1502):167–98.
13. Prudnikov AP, Brychkov YuA, Marichev OI. *Integrals and Series. Special Functions*. Moscow: Nauka; 1983.
14. Liu GR. A combined finite element/strip element method for analyzing elastic wave scattering by cracks and inclusions in laminates. *Comput Mech* 2002;**28**:76–81.

Translated by P.S.C.